Hybridizable interior penalty methods for the class of non-uniform second-order elliptic problems

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Introduction

Advection-Diffusion-Reaction problems are relevant in (Computational) Geosciences:

$$\nabla \cdot (-\boldsymbol{\kappa} \boldsymbol{\nabla} u + \boldsymbol{\beta} u) + \gamma u = f \quad \text{in} \quad \Omega \subset \mathbb{R}^d, \tag{1}$$

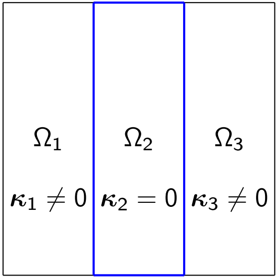
where u denotes the state variable.

1. A wide range of physical processes and mathematical challenges :
   - Pure diffusion problem with $\boldsymbol{\beta} = 0$ (Darcy' or Fick' laws),
   - Pure (linear) advection problem with $\boldsymbol{\kappa} = 0$ (Neutron transport),
   - Mixed ADR problem w.r.t. $0 < \mathrm{Pe} < \infty$ (contaminant transport).

In all these situations, mathematical nature is uniform on the whole domain Ω.

2. Here, we focus on non-uniform second-order elliptic problems :

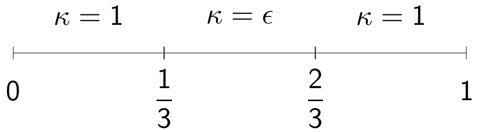
- Non-uniformity = Hyperbolic in a subpart of Ω and Elliptic in the rest



- Heat and Mass transfer in fractured porous media

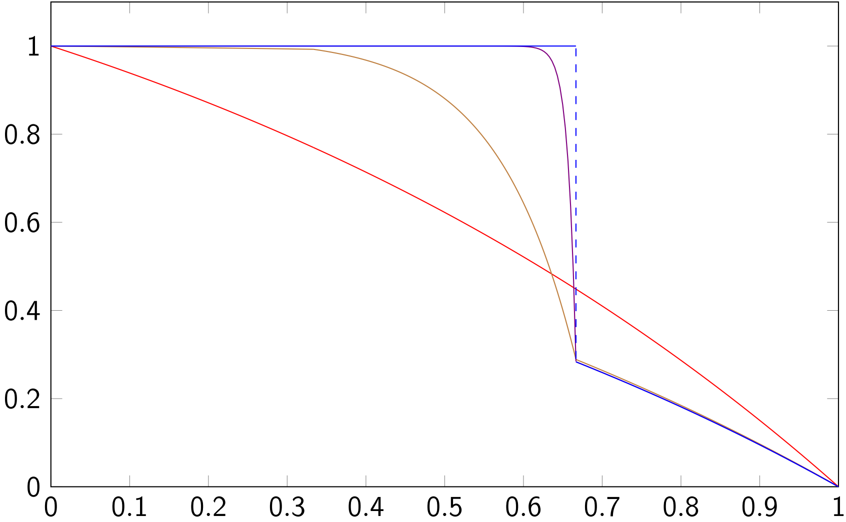
Toy model problem (1D) :

Consider $(-\kappa u_\epsilon' + u_\epsilon)' = 0 \text{ in }]0,1[$ with Dirichlet B.C. $u_\epsilon(0) = 1$ and $u_\epsilon(1) = 0$:



We represent the analytical solution $u_\epsilon(x)$ with $\lim \epsilon \to 0$.

Example 1D : Analytical solution $u_\epsilon(x)$ with $\epsilon = \{1, 0.1, 0.01, \ldots, 0\}$.



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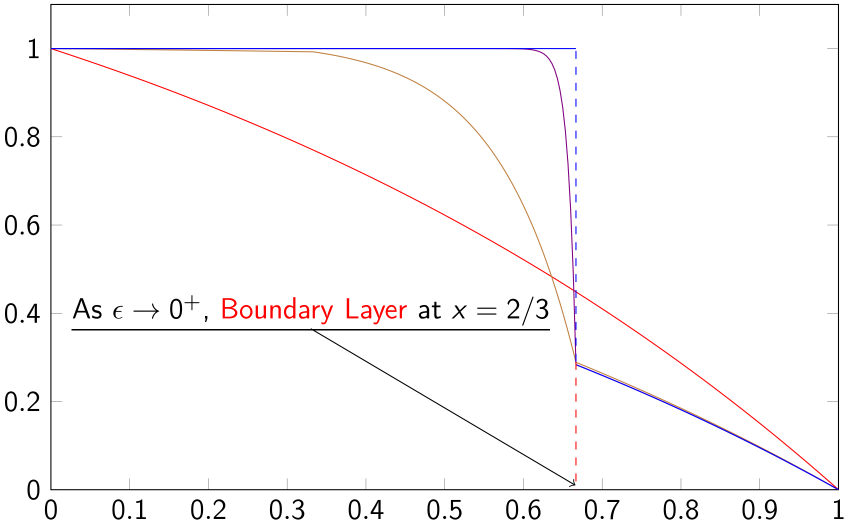


Motivations & Outline

Main goal : an unified Hybridizable DG method for solving large class of problems

- The continuous problem (sense of Fichera)
- Hybridizable IP approximations
- Numerical experiments

Following Fichera (1956):

The Dirichlet boundary value problem reads:

$$\begin{array}{rcll} \nabla \cdot (-\boldsymbol{\kappa}\nabla u + \boldsymbol{\beta} u) + \mu u & = & f & \text{in} \quad \Omega \backslash \mathcal{I}_{\boldsymbol{\kappa}}, \\ [\![-\boldsymbol{\kappa}\nabla u + \boldsymbol{\beta} u]\!] & = & 0 & \text{on} \quad \mathcal{I}_{\boldsymbol{\kappa}}, \\ [\![u]\!] & = & 0 & \text{on} \quad \mathcal{I}^{+}_{\boldsymbol{\kappa},\boldsymbol{\beta}}, \\ u & = & g & \text{on} \quad \Gamma^{-}_{\boldsymbol{\kappa},\boldsymbol{\beta}}, \end{array} \tag{2}$$

where $[\![\cdot]\!]$ denotes the normal jump operator and,

- $\mathcal{I}_{\boldsymbol{\kappa}} := \{x \in \Omega \, : \, \partial\Omega^{\mathrm{hyp}}_{\boldsymbol{\kappa}} \cap \partial\Omega^{\mathrm{ell}}_{\boldsymbol{\kappa}}\}$ and $\mathcal{I}^{\pm}_{\boldsymbol{\kappa},\boldsymbol{\beta}} := \{x \in \mathcal{I}_{\boldsymbol{\kappa}} \, : \, \pm\boldsymbol{\beta}(x) \cdot \boldsymbol{n}_I > 0\}$
- $\Gamma^{-}_{\boldsymbol{\kappa},\boldsymbol{\beta}}$ corresponds to the non-degenerative inflow boundary :

$$\Gamma^{-}_{\boldsymbol{\kappa},\boldsymbol{\beta}} := \{x \in \partial\Omega \, : \, \boldsymbol{n}^t\boldsymbol{\kappa}(x)\boldsymbol{n} > 0 \quad \text{or} \quad \boldsymbol{\beta}\cdot\boldsymbol{n} < 0\},$$

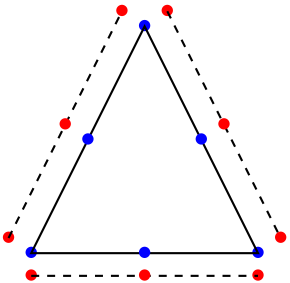
and $\boldsymbol{n}$ denotes the unit outward normal to $\partial\Omega$.

Hybridizable Interior Penalty method

- HIP method = Nonconforming method

Term *nonconforming* = no-regularity assumptions on discrete variables.

- The scalar variable u is approximated locally by
  - Interior variable u_h defined in each element A of $\mathcal{T}_h$: $u_h \in \mathbb{P}_k(A)$
  - Trace variable $\hat{u}_h$ defined on each face F of the mesh skeleton: $\hat{u}_h \in \mathbb{P}_k(F)$



Intuitive derivation

- Weak formulation on $A \in \mathcal{T}_h$ (Local problem)

$$-(\boldsymbol{\sigma}_h, \nabla v_h)_A + \langle \hat{\boldsymbol{\sigma}}_h(u_h, \hat{u}_h) \cdot \boldsymbol{n}, v_h \rangle_{\partial A} + (\mu u_h, v_h)_A = (f, v_h)_A \tag{3}$$

- By summing-up overall element $A \in \mathcal{T}_h$ (Global problem)

$$-(\boldsymbol{\sigma}_h, \nabla v_h)_\Omega + (\mu u_h, v_h)_\Omega + \sum_{A \in \mathcal{T}_h} \langle \hat{\boldsymbol{\sigma}}_h(u_h, \hat{u}_h) \cdot \boldsymbol{n}, v_h \rangle_{\partial A} = (f, v_h)_\Omega \tag{4}$$

- Continuity of the normal flux (Closure equation)

$$\sum_{A \in \mathcal{T}_h} \langle \hat{\boldsymbol{\sigma}}_h(u_h, \hat{u}_h) \cdot \boldsymbol{n}, \hat{v}_h \rangle_{\partial A} = 0 \tag{5}$$

- By combining these equations

$$-(\boldsymbol{\sigma}_h, \nabla v_h)_\Omega + (\mu u_h, v_h)_\Omega + \sum_{A \in \mathcal{T}_h} \langle \hat{\boldsymbol{\sigma}}_h(u_h, \hat{u}_h) \cdot \boldsymbol{n}, v_h - \hat{v}_h \rangle_{\partial A} = (f, v_h)_\Omega \tag{6}$$

Numerical fluxes of HDG

Following Cockburn (2009) : $\hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n} = \boldsymbol{\sigma}_h \cdot \boldsymbol{n} + \tau(u_h - \hat{u}_h)$

The role of τ :

- Ensure that local and global solvers are well-posed,
- Transform the discrepancy $(u_h - \hat{u}_h)$ on ∂A into a *positive energy*,
- Enhance the stability of the HDG method.

Mathematical expressions

- For $e \in \mathcal{F}_h^{\mathfrak{i}}$:

$$\hat{u}_h = \{u_h\}_{\boldsymbol{\omega}} + \alpha_e[\![\boldsymbol{\sigma}_h]\!] \quad \text{and} \quad \hat{\boldsymbol{\sigma}}_h = \{\boldsymbol{\sigma}_h\}_{\overline{\boldsymbol{\omega}}} + \gamma_e[\![u_h]\!] \tag{7}$$

  where $\alpha_e(\tau)$, $\gamma_e(\tau)$, $\boldsymbol{\omega}(\tau)$ et $\{\cdot\}_{\boldsymbol{\omega}}$, $\{\cdot\}_{\overline{\boldsymbol{\omega}}}$: traces operators.
- For $e \in \mathcal{F}_h^{\mathfrak{b}}$:

$$\hat{u}_h = g, \quad \text{and} \quad \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}_e = \boldsymbol{\sigma}_h \cdot \boldsymbol{n}_e + \gamma_e(u_h - g). \tag{8}$$

How to choose Stabilization Parameter τ_e

- Pure diffusive regime ($\boldsymbol{\beta} = \mathbf{0}$) :

$$\tau_e^d := \frac{\alpha_0(k+1)(k+d)\kappa_e}{|e|} \quad \text{where} \quad \kappa_e = \boldsymbol{n}_e \cdot \boldsymbol{\kappa} \cdot \boldsymbol{n}_e$$

  Here κ_e is the normal diffusivity at the interface e.

- Pure advective regime ($\boldsymbol{\kappa} = \mathbf{0}$) :

$$\tau_e^{\mathrm{up}} := \max(0, -\beta_e) \quad \text{where} \quad \beta_e = \boldsymbol{\beta} \cdot \boldsymbol{n}_e.$$

  Here τ_e^{up} characterized the traditional *upwind-strategy*

- Intermediate regime :

$$\tau_e = \tau_e^d + \tau_e^{\mathrm{up}}.$$

Discrete weak problem

Compact HIP formulation:

Find $(u_h, \hat{u}_h) \in V_h \times \hat{V}_h^g$ such that

$$a_h^{(\epsilon)}(u_h, \hat{u}_h; v_h, \hat{v}_h) = l(v_h) \qquad \forall (v_h, \hat{v}_h) \in V_h \times \hat{V}_h^0$$

where $\epsilon \in \{0, \pm 1\}$ and the bilinear form $a_h^{(\epsilon)}$ is given by

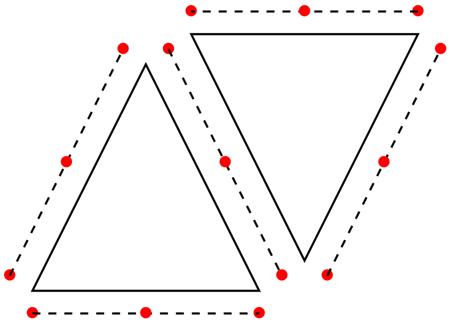
$$\begin{aligned} a_h^{(\epsilon)}(u_h, \hat{u}_h; v_h, \hat{v}_h) := \; & (\boldsymbol{\kappa} \nabla u_h, \nabla v_h)_\Omega - (\boldsymbol{\beta} u_h, \nabla v_h)_\Omega + (\mu u_h, v_h)_\Omega \\ & - \sum_{A \in \mathcal{T}_h} \langle \boldsymbol{\kappa} \nabla u_h \cdot \boldsymbol{n} - \boldsymbol{\beta} \cdot \boldsymbol{n} u_h, v_h - \hat{v}_h \rangle_{\partial A} \\ & - \sum_{A \in \mathcal{T}_h} \epsilon \langle \boldsymbol{\kappa} \nabla v_h \cdot \boldsymbol{n}, u_h - \hat{u}_h \rangle_{\partial A} \\ & + \sum_{A \in \mathcal{T}_h} \langle \tau (u_h - \hat{u}_h), v_h - \hat{v}_h \rangle_{\partial A} \\ & + \sum_{F \in \Gamma^+_{\boldsymbol{\kappa}, \boldsymbol{\beta}}} \langle \boldsymbol{\beta} \cdot \boldsymbol{n} \hat{u}_h, \hat{v}_h \rangle_F \end{aligned}$$

- Three HIP variants : H-IIP ($\epsilon = 0$), H-SIP ($\epsilon = +1$), and H-NIP ($\epsilon = -1$)

Static condensation: $a_h^{(\epsilon)}(u_h, \hat{u}_h; v_h, 0)$ and $a_h^{(\epsilon)}(u_h, \hat{u}_h; 0, \hat{v}_h)$

Algebraic matrix equation:

$$\begin{bmatrix} \mathbb{A}_{uu} & \mathbb{A}_{u\hat{u}} \\ \mathbb{A}_{\hat{u}u} & \mathbb{A}_{\hat{u}\hat{u}} \end{bmatrix} \cdot \begin{bmatrix} U_h \\ \hat{U}_h \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix} \tag{9}$$



- Step 1. Elimination of interior variable u_h :

$$U_h = \mathbb{A}_{uu}^{-1} F - \mathbb{A}_{uu}^{-1} \mathbb{A}_{u\hat{u}} \hat{U}_h$$

- Step 2. Lagrange multipliers system:

$$\mathbb{A} \hat{U}_h = b$$

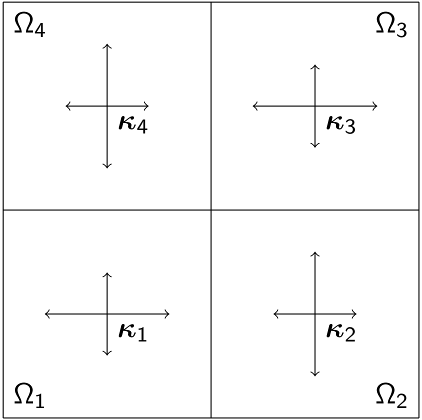
- Step 3. Reconstruction of u_h, $\boldsymbol{\sigma}_h$ and $\hat{\boldsymbol{\sigma}}_h$

Numerical experiments

The proposed HIP method is implemented via the NGsolve library :

- Pure diffusion problems with low and high regularity solution
- Pure advection problems
- Mixed ADR problems with high Péclet number
- Degenerate ADR problems

Test 1: Pure diffusive problem (smooth solution)



- $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 = [0,1]^2$,
- Heterogeneous/Anisotropic diffusion tensor $\boldsymbol{\kappa}$:

$$\boldsymbol{\kappa}_{1,3}(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix} \qquad \text{for } (x,y) \in \Omega_{1,3},$$

$$\boldsymbol{\kappa}_{2,4}(x,y) = \begin{bmatrix} \epsilon & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{for } (x,y) \in \Omega_{2,4}.$$

  with $\varepsilon \in \{1 \ , \ 10 \text{ and } 100\}$.
- Exact solution in Ω : $u(x,y) = \sin(\pi x)\sin(\pi y)$

Test 1: Pure diffusive problem (smooth solution)

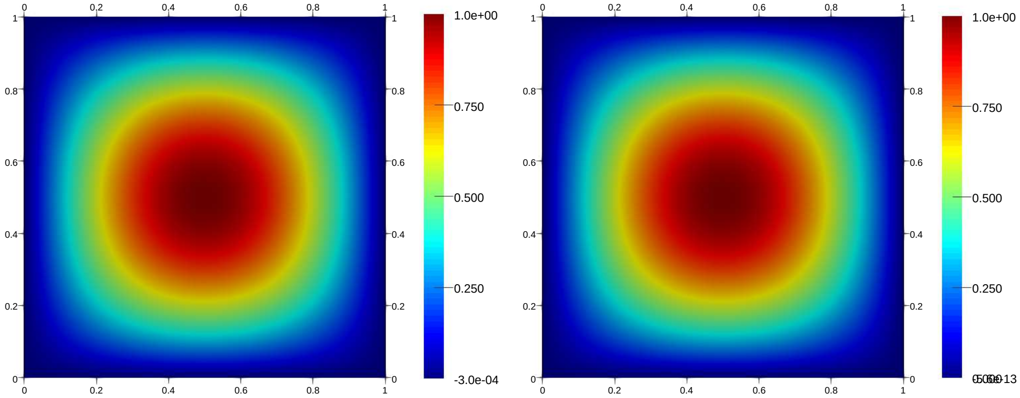


Figure: Representation of $u_h \in \mathbb{P}_k(\mathcal{T}_h)$ with $k = 1$ (left) and $k = 5$ (right).

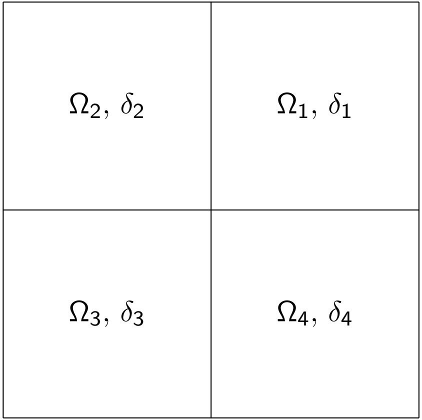
Test 1: Pure diffusive problem (smooth solution)

		$\epsilon = 1$				$\epsilon = 10$				$\epsilon = 100$			
		$\Vert u - u_h \Vert_{L_2}$		$\Vert u - u_h \Vert_{H^1}$		$\Vert u - u_h \Vert_{L_2}$		$\Vert u - u_h \Vert_{H^1}$		$\Vert u - u_h \Vert_{L_2}$		$\Vert u - u_h \Vert_{H^1}$	
k	h	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate
	4	4.8e-02	-	8.3e-01	-	4.4e-02	-	8.2e-01	-	4.2e-02	-	8.2e-01	-
	8	1.2e-02	2.0	4.1e-01	1.0	1.1e-02	2.0	4.1e-01	1.0	1.1e-02	1.9	4.3e-01	0.9
1	16	2.9e-03	2.0	2.0e-01	1.0	2.7e-03	2.0	2.0e-01	1.0	2.8e-03	2.0	2.1e-01	1.0
	32	7.3e-04	2.0	1.0e-01	1.0	6.6e-04	2.0	1.1e-01	1.0	6.8e-04	2.0	1.1e-01	1.0
	64	1.8e-04	2.0	5.0e-02	1.0	1.6e-04	2.0	5.0e-02	1.0	1.6e-04	2.0	5.2e-02	1.0
	4	4.5e-03	-	1.5e-01	-	4.6e-03	-	1.5e-01	-	4.7e-03	-	1.6e-01	-
	8	5.8e-04	3.0	3.8e-02	2.0	6.0e-04	2.9	3.8e-02	2.0	6.7e-04	2.8	4.1e-02	2.0
2	16	7.3e-05	3.0	9.4e-03	2.0	7.6e-05	3.0	9.4e-03	2.0	9.0e-05	2.9	1.0e-02	2.0
	32	9.2e-06	3.0	2.3e-03	2.0	9.7e-06	3.0	2.3e-03	2.0	1.1e-05	3.0	2.5e-03	2.0
	64	1.1e-06	3.0	5.8e-04	2.0	1.2e-06	3.0	5.8e-04	2.0	1.4e-06	3.0	6.0e-04	2.0
	4	4.9e-04	-	1.9e-02	-	5.0e-04	-	2.0e-02	-	5.5e-04	-	2.2e-02	-
	8	2.9e-05	4.0	2.3e-03	3.0	3.1e-05	4.0	2.4e-03	3.0	3.5e-05	4.0	2.6e-03	3.1
3	16	1.8e-06	4.0	2.9e-04	3.0	1.8e-06	4.1	2.9e-04	3.0	2.1e-06	4.0	3.1e-04	3.0
	32	1.1e-07	4.0	3.6e-05	3.0	1.1e-07	4.0	3.6e-05	3.0	1.3e-07	4.0	3.8e-05	3.0
	64	6.8e-09	4.0	4.5e-06	3.0	6.8e-09	4.0	4.5e-06	3.0	8.0e-09	4.0	4.7e-06	3.0

Table: History of convergence for the smooth diffusive solution test case.

Test 2: Pure diffusive problem (rough solution)

(Rivière, 2008)



- Homogeneous diffusion tensor $\boldsymbol{\kappa} = \delta \boldsymbol{I_d}$, with $\delta_1 = \delta_3 = 5$ and $\delta_2 = \delta_4 = 1$.
- Exact solution in polar coordinates :

$$u_{|\Omega_i}(r,\theta) = r^{\alpha}(a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta)) \quad \text{in } \Omega_i,$$

where $\alpha = 0.5354409456$ and

$$\begin{aligned} a_1 &= 0.4472135955 & b_1 &= 1 \\ a_2 &= -0.7453559925 & b_2 &= 2.333333333 \\ a_3 &= -0.9441175905 & b_3 &= 0.55555555555 \\ a_4 &= -2.401702643 & b_4 &= -0.4814814814 \end{aligned}$$

Test 2: Pure diffusive problem (rough solution)

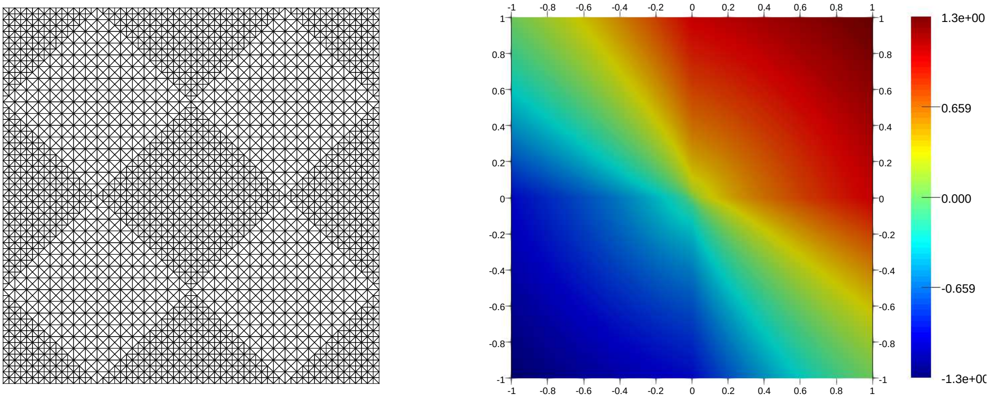


Figure: Representation of the mesh (left) and the analytical solution (right).

Test 2: Pure diffusive problem (rough solution)

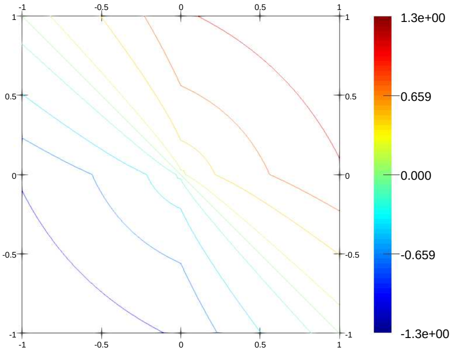


Figure: u_h with $k = 1$ and $h = 1/64$.

Estimated convergence rates :

$$\| u - u_h \|_{L_2} \leq O(h^{2\alpha}) \qquad \| u - u_h \|_{H^1} \leq O(h^{\alpha})$$

	$\alpha = 0.5354409456$			
	$\| u - u_h \|_{L_2}$		$\| u - u_h \|_{H^1}$	
h	Error	Rate	Error	Rate
4	1.1e-01	-	7.8e-01	-
8	4.3e-02	1.303	4.8e-01	0.697
16	1.8e-02	1.226	3.2e-01	0.588
32	8.1e-03	1.200	2.2e-01	0.568
64	3.6e-03	1.150	1.5e-01	0.556

Table: History of convergence for $k = 1$.

Test 3: Pure advective problems (discontinuous solution)

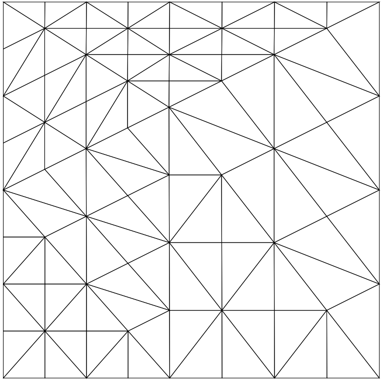


Figure: Partition $\mathcal{T}_h$ for $h = 1/8$

- Dirichlet BC imposed only inflow part of $\partial\Omega$,
- No diffusion $\boldsymbol{\kappa} = \mathbf{0}$,
- Uniform Velocity field $\boldsymbol{\beta} = [\beta_1, \beta_2] \equiv [2, 1]$,
- Exact solution in Ω

$$u(x, y) = \begin{cases} 1 & \text{if } y > 0.5(1 + x), \\ 0 & \text{else} \end{cases}$$

Test 3: Pure advective problem (discontinuous solution)

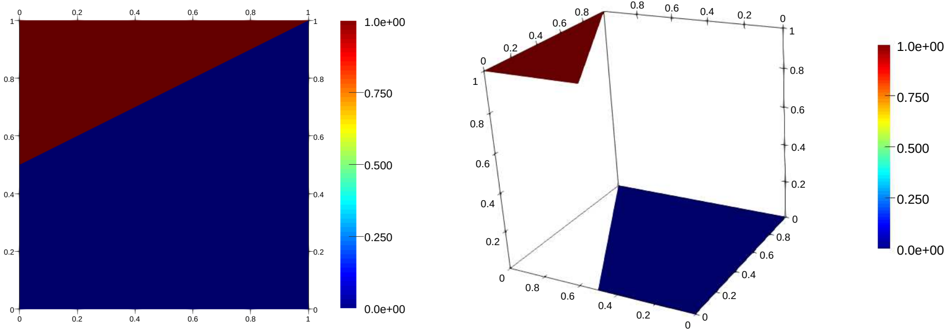
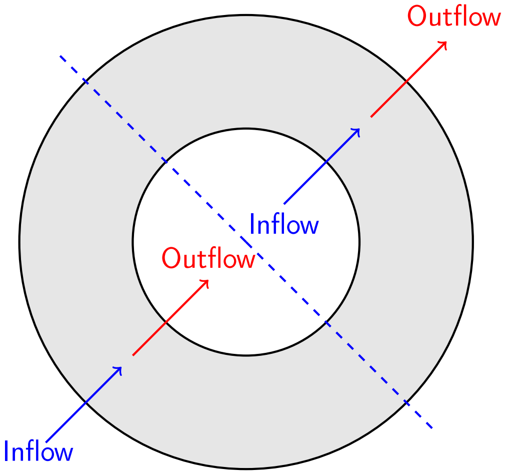


Figure: Representation of the numerical solution for the advective problem for $k = 1$ and $h = 1/8$.

Test 4: Pure advective problem



- Dirichlet BC imposed only inflow part of $\partial\Omega$,
- No diffusion $\boldsymbol{\kappa} = \mathbf{0}$,
- Uniform Velocity field $\boldsymbol{\beta} = [\beta_1, \beta_2] \equiv [1, 1]$
- Exact solution in Ω :

$$u(x, y) = \exp[\cos(x - y)]$$

Test 4: Pure advective problem

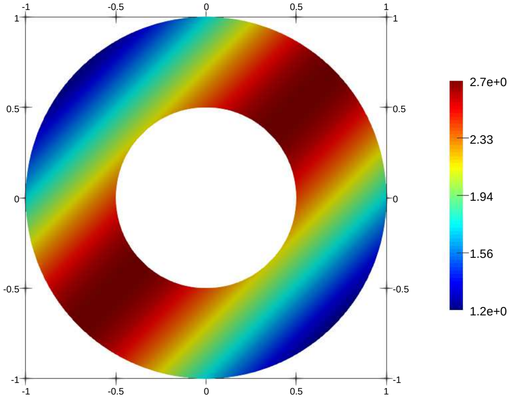


Figure: Representation of the numerical solution for the advective rotating test case for $k = 1$ and $h = 1/64$.

Test 4: Purely convective rotative

		HDG								
h^{-1}	k	$\| u - u_h \|_{L_2}$ Erreur	Ordre	$\| u - u_h \|_{H^1}$ Erreur	Ordre	k	$\| u - u_h \|_{L_2}$ Erreur	Ordre	$\| u - u_h \|_{H^1}$ Erreur	Ordre
4		1.5e-02	-	3.2e-01	-		6.4e-04	-	3.3e-02	-
8		5.8e-03	1.4	1.9e-01	0.7		2.9e-04	1.1	1.6e-02	1.1
16	1	1.5e-03	2.0	9.7e-02	1.0	2	3.8e-05	2.9	3.8e-03	2.0
32		3.8e-04	2.0	4.8e-02	1.0		4.9e-06	3.0	9.5e-04	2.0
64		9.5e-05	2.0	2.4e-02	1.0		6.0e-07	3.0	2.4e-04	2.0
4		3.5e-05	-	2.2e-03	-		2.5e-06	-	1.6e-04	-
8		8.6e-06	2.0	6.8e-04	1.7		3.8e-07	2.7	3.4e-05	2.2
16	3	5.9e-07	3.9	8.9e-05	2.9	4	1.3e-08	4.9	2.2e-06	4.0
32		3.5e-08	4.1	1.1e-05	3.0		3.9e-10	5.0	1.4e-07	4.0
64		2.2e-09	4.0	1.4e-06	3.0		1.2e-11	5.0	8.6e-09	4.0

Table: History of convergence for the convection test case.

Test 5: Mixed Advection-Diffusion problem (high Péclet)

(Egger and Schöberl, 2009)

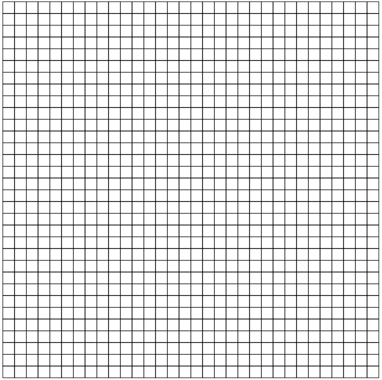


Figure: Square mesh for $h = 1/64$

- Diffusion tensor $\boldsymbol{\kappa} = \delta \boldsymbol{I_d}$ with $\delta = 1e - 3$.
- Uniform Velocity field $\boldsymbol{\beta} = [\beta_1, \beta_2] \equiv [2, 1]$
- Exact solution in Ω

$$u(x, y) = \left(x - \frac{e^{\beta_1 x/\delta} - 1}{e^{\beta_1/\delta} - 1} \right) \cdot \left(y - \frac{e^{\beta_2 y/\delta} - 1}{e^{\beta_2/\delta} - 1} \right)$$

Test 5: Mixed Advection-Diffusion problem (high Péclet)

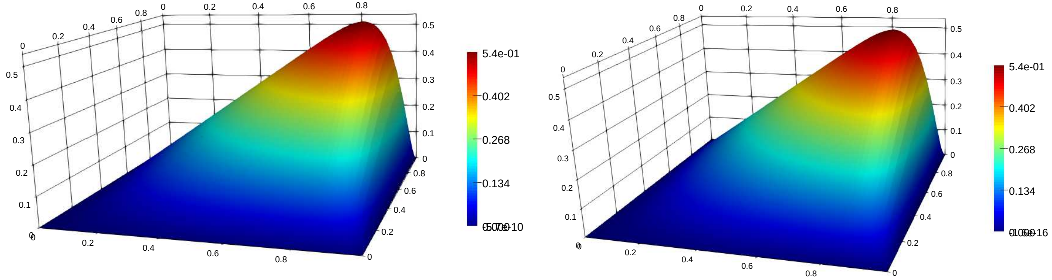


Figure: Numerical solution for $k = 1$ (left) vs Analytical solution (right) in 3d.

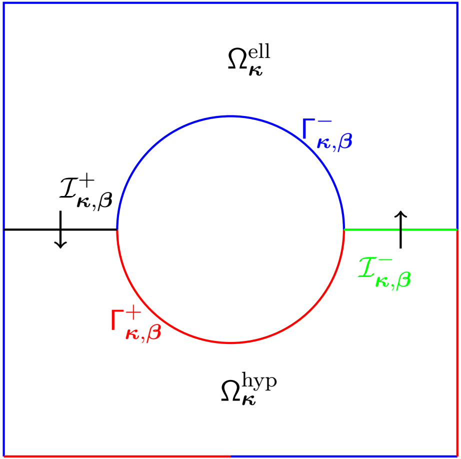
Test 5: Mixed Advection-Diffusion problem (high Péclet)

		HDG								
h^{-1}	k	$\| u - u_h \|_{L_2}$ Erreur	Ordre	$\| u - u_h \|_{H^1}$ Erreur	Ordre	k	$\| u - u_h \|_{L_2}$ Erreur	Ordre	$\| u - u_h \|_{H^1}$ Erreur	Ordre
4		7.4e-02	-	1.2e+00	-		1.9e-02	-	5.9e-01	-
8		2.4e-02	1.6	9.0e-01	0.5		3.8e-03	2.4	2.5e-01	1.2
16	1	6.3e-03	1.9	5.2e-01	0.8	2	5.8e-04	2.7	7.8e-02	1.7
32		1.6e-03	2.0	2.7e-01	0.9		7.6e-05	2.9	2.1e-02	1.9
64		3.9e-04	2.0	1.4e-01	1.0		9.7e-06	3.0	5.3e-03	2.0
4		4.7e-03	-	2.1e-01	-		1.0e-03	-	6.2e-02	-
8		5.1e-04	3.2	5.0e-02	2.1		5.9e-05	4.1	7.7e-03	3.0
16	3	3.9e-05	3.7	8.0e-03	2.6	4	2.3e-06	4.7	6.7e-04	3.5
32		2.6e-06	3.9	1.1e-03	2.9		7.8e-08	4.9	2.4e-04	1.5
64		1.7e-07	4.0	2.7e-04	2.0		2.5e-09	5.0	2.3e-04	0.0
4		1.9e-04	-	1.5e-02	-		3.2e-05	-	3.0e-03	-
8		5.8e-06	5.1	9.8e-04	3.9		5.0e-07	6.0	2.5e-04	3.6
16	5	1.2e-07	5.6	2.4e-04	2.1	6	5.1e-09	6.6	2.3e-04	0.1
32		2.0e-09	5.9	2.3e-04	0.0		5.2e-11	6.6	2.3e-04	0.0

Table: History of convergence for the advection diffusion test case.

Test 6: Degenerate ADR problem

(Di Pietro, Droniou and Ern, 2009)



- Homogeneous diffusion tensor $\boldsymbol{\kappa} = \delta \boldsymbol{I_d}$,

$$\delta(r,\theta) = \begin{cases} \pi & \text{si } 0 \leq \theta \geq \pi, \\ 0 & \text{si } 0 \leq \theta \leq \pi, \end{cases}$$

- Uniform Velocity field $\boldsymbol{\beta} = \frac{\overrightarrow{e_\theta}}{r}$
- Uniform reaction coefficient $\mu = 10^{-6}$.
- Exact solution in polar coordinates

$$u(r,\theta) = \begin{cases} (\theta-\pi)^2 & \text{si } 0 \leq \theta \leq \pi, \\ 3\pi(\theta-\pi) & \text{si } 0 \leq \theta \leq \pi, \end{cases} \quad ,$$

Test 6: Degenerate ADR problem

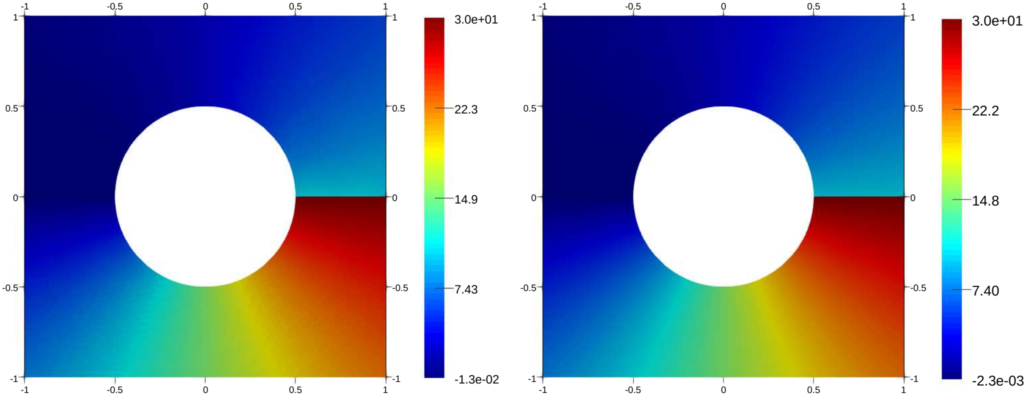


Figure: Numerical solution $k = 0$ (left) vs Analytical solution (right).

Test 6: Degenerate ADR problem

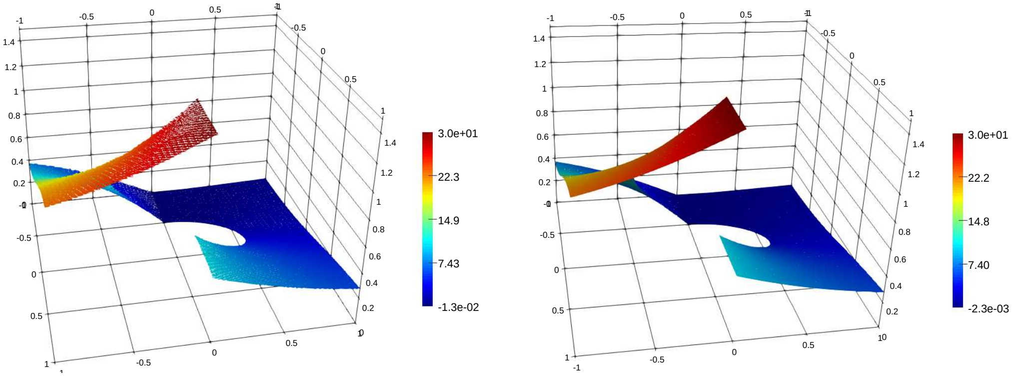


Figure: 3D Representation / Numerical solution $k = 0$ (left) vs Analytical solution (right).

Test 6: Degenerate ADR problem

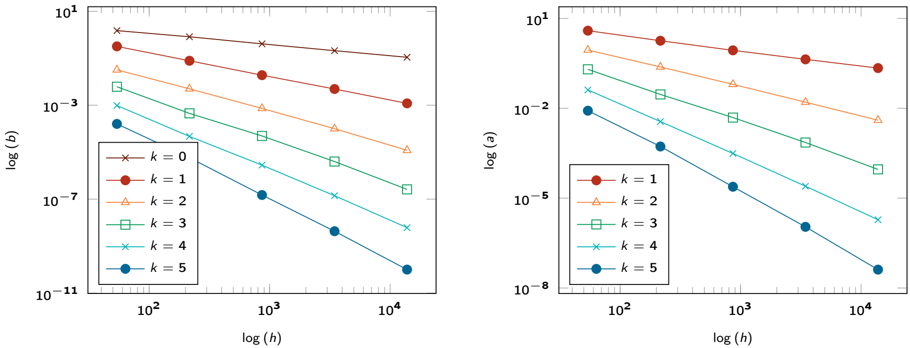


Figure: History of convergence error in the L_2-norm (left) and H^1-norm (right).

Conclusion

Unified Hybridizable IP method for the class of degenerate ADR problem :

- High-Order DG approximation with $k \geq 0$
- Highly parallelizable (static condensation)
- Compact stencil (face localization of dofs)
- Optimal convergence rates for all regimes

To explore :

- $u_h^* =$ Post-processing of $(u_h, \hat{u}_h)$:

$$\| u - u_h^* \|_{\mathrm{L}_2} \leq O(h^{k+2}) \qquad \text{(superconvergence)}$$

- $\boldsymbol{\sigma}_h^* =$ H(div)-conform reconstruction of $(\boldsymbol{\sigma}_h, \hat{\boldsymbol{\sigma}}_h)$:

$$\| \boldsymbol{\sigma} - \boldsymbol{\sigma}_h^* \|_{\mathrm{div}} \leq O(h^{k+1})$$